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Journal of Approximation Theory 135 (2005) 160–175
Journal of Approximation Theory
www.elsevier.com/locate/jat

Asymptotic formulae of Mehler–Heine-type for certain classical polyorthogonal polynomials

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Received 6 October 2003; received in revised form 15 February 2005; accepted 13 April 2005

Communicated by Guillermo López Lagomasino
Available online 14 June 2005

Abstract

The purpose of this article is to give some asymptotic formulae of polyorthogonal polynomials with respect to some classical measures. The formulae are analogous to the Mehler–Heine formulae for Jacobi and Laguerre polynomials.

Keywords: Orthogonal polynomials; Polyorthogonal polynomials; Multiple orthogonal polynomials; Mehler–Heine-type formulae

1. Introduction

The denominators of the Padé approximants of the Stieltjes transform of a positive measure μ coincide with the orthogonal polynomials associated with μ. Similarly, the Stieltjes transforms of several positive measures $\mu_1, \mu_2, \ldots, \mu_l$ can be approximated by rational functions which are called the simultaneous Padé approximants. The denominators of these approximants have some orthogonality properties. They are called polyorthogonal polynomials (of type II) or multiple orthogonal polynomials (see [2,6] or [11]).

In this article, we shall consider the polyorthogonal polynomials of type II which are called Jacobi–Jacobi (J–J) and Jacobi–Laguerre (J–L) type. The J–J type polynomials are considered by Kalyagin [5] and the J–L type polynomials are considered by Sorokin [7].

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[1] Supported by JSPS Fellowship for Young Scientists.

doi:10.1016/j.jat.2005.04.005

The definition of them (we give in §3) are similar to the Jacobi and Laguerre polynomials. Thus they are expected to have similar properties as those of the Jacobi and Laguerre polynomials. Especially, we are interested in asymptotic properties when the degrees of these polynomials go to infinity. The main purpose of the present paper is to give new asymptotic formulae, which are analogous to the Mehler–Heine-type asymptotic formulae of Jacobi and Laguerre polynomials. Other various properties of these polynomials are well studied in Sections 3.5 and 3.6 of [10].

Several strong results have been obtained dealing with Mehler–Heine-type formulae of orthogonal polynomials in recent years [1,9]. But the known Mehler–Heine-type formulae of polyorthogonal polynomials are only a few [3].

To prove the Mehler–Heine-type formulae for Jacobi and Laguerre polynomials usually the explicit representations of these polynomials are used (see [8, §8.1]). Our polynomials also have similar explicit representations, but they are so complicated that it is difficult to apply the same method as Jacobi and Laguerre polynomials to our case. Therefore, we shall give new certain different representations using the Jacobi and Laguerre polynomials. Then we shall prove the desired asymptotic formulae from the representations by using the Mehler–Heine-type formulae for the Jacobi and Laguerre polynomials.

2. Preliminary

In this section we fix the notations and terminology we use throughout this paper.

The Borel σ-algebra on the real line is the smallest σ-algebra containing all compact subsets of $\mathbf{R}$. We say that a σ-additive set function defined on the Borel σ-algebra is a *finite positive Borel measure* if it takes finite nonnegative values. A point $\lambda \in \mathbf{R}$ is called a *point of increase* of the measure μ if $\mu(\lambda - \varepsilon, \lambda + \varepsilon) > 0$ for every $\varepsilon > 0$. The set of points of increase of μ is called the spectrum of μ. The *support* of μ is the smallest interval containing the spectrum of μ.

We use the following notation:

$$f(n) = \mathcal{O}(g(n)) \quad \text{as } n \to \infty \tag{1}$$

to state that $f(n)/g(n)$ is bounded as $n \to \infty$.

3. Certain Angelesco systems (see [6])

Let us consider the following Stieltjes integrals:

$$\omega_i(z) = \int_{\Delta_i} \frac{\mu_i(d\zeta)}{z - \zeta}, \quad 1 \leqslant i \leqslant l. \tag{2}$$

Set $\Delta_i \subset \mathbf{R}$ and μ_i is a finite positive Borel measure with support Δ_i for each i. For a vector-index $\boldsymbol{n} = (n_1, n_2, \ldots, n_l) \in \mathbf{Z}_+^l$, there exists a polynomial $p = p_{\boldsymbol{n}}$ with deg $p \leqslant |\boldsymbol{n}|$ such that

$$\int_{\Delta_i} \zeta^\nu p(\zeta)\mu_i(d\zeta) = 0, \quad 0 \leqslant \nu \leqslant n_i - 1 \tag{3}$$

for $1 \leqslant i \leqslant l$. Because of this orthogonality, we call $p_{\boldsymbol{n}}$ an $\boldsymbol{n}$th polyorthogonal (precisely, l-orthogonal) polynomial (of type II).

An index $\boldsymbol{n}$ is said to be normal if $\deg p_{\boldsymbol{n}} = |\boldsymbol{n}|$ for any $\boldsymbol{n}$th Padé pair. And if all $\boldsymbol{n}$ are normal, $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_l)$ is called a complete system.

We present an example of a complete system, which is called an Angelesco system. Suppose that the spectrum of each μ_i is infinite and $\{\Delta_i\}$ are pairwise disjoint. Then for any index $\boldsymbol{n}$, the $\boldsymbol{n}$th l-orthogonal polynomial p has n_i simple zeros inside Δ_i for $1 \leqslant i \leqslant l$. Consequently $\boldsymbol{\omega}$ is complete.

Especially, let us consider the following two Angelesco systems:

(1) $l = 2$, $\Delta_1 = [-1, +1]$, $\Delta_2 = [+1, a]$, $(a > 1)$ and

$$d\mu_1 = |h(x)|\,dx \quad \text{on } \Delta_1 \quad \text{and} \quad d\mu_2 = |h(x)|\,dx \quad \text{on } \Delta_2, \tag{4}$$

where $h(x) = (1-x)^\alpha (1+x)^\beta (a-x)^\gamma$ for $\alpha, \beta, \gamma > -1$.

(2) $l = 2$, $\Delta_1 = [0, \infty)$, $\Delta_2 = [b, 0]$, $(b < 0)$ and

$$d\mu_1 = |h(x)|\,dx \quad \text{on } \Delta_1 \quad \text{and} \quad d\mu_2 = |h(x)|\,dx \quad \text{on } \Delta_2, \tag{5}$$

where $h(x) = e^{-x} x^\alpha (x-b)^\beta$ for $\alpha, \beta > -1$.

These systems define the 2-orthogonal polynomials $p^{(\alpha,\beta,\gamma)}_{(m,n)}$ and $l^{(\alpha,\beta)}_{(m,n)}$, respectively. These polynomials are sometimes called J–J and J–L polynomials. We note J–L polynomials can be constructed as a limit case of J–J polynomials (see [10, Section 3.5]). Furthermore, we set

$$p_n^{(\alpha,\beta,\gamma)} = \begin{cases} p^{(\alpha,\beta,\gamma)}_{(k,k)} & \text{if } n = 2k, \\ p^{(\alpha,\beta,\gamma)}_{(k+1,k)} & \text{if } n = 2k+1, \end{cases}, \quad l_n^{(\alpha,\beta)} = \begin{cases} l^{(\alpha,\beta)}_{(k,k)} & \text{if } n = 2k, \\ l^{(\alpha,\beta)}_{(k+1,k)} & \text{if } n = 2k+1. \end{cases}.$$

In this article we are interested in asymptotic properties of these polynomials.

4. The Mehler–Heine formulae

We recall some well-known and important properties of the Jacobi and Laguerre polynomials in this section (see [8]).

For $\alpha, \beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by

$$\begin{aligned}(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) \\ = \binom{n+\alpha}{n}^{-1} \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.\end{aligned} \tag{6}$$

Using Leibniz' formula, we can show $P_n^{(\alpha,\beta)}(x)$ are polynomials of degree n. And they have the orthogonality conditions

$$\int_{-1}^{+1} x^\nu P_n^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta\,dx = 0, \quad 0 \leqslant \nu \leqslant n-1. \tag{7}$$

Conversely, (7) and $P_n^{(\alpha,\beta)}(1) = 1$ imply (6). The Jacobi polynomials have the explicit representation:

$$\frac{P_n^{(\alpha,\beta)}(x)}{\Gamma(\alpha+1)} = \sum_{v=0}^{n} \frac{n!}{(n-v)!} \frac{\Gamma(n+\alpha+\beta+v+1)}{\Gamma(n+\alpha+\beta+1)} \frac{1}{\Gamma(\alpha+v+1)} \left(\frac{x-1}{2}\right)^v. \tag{8}$$

Similarly, for $\alpha > -1$ the Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by

$$e^{-x} x^{\alpha} L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}^{-1} \frac{1}{n!} \left(\frac{d}{dx}\right)^n (e^{-x} x^{n+\alpha}). \tag{9}$$

$L_n^{(\alpha)}(x)$ are polynomials of degree n and have the orthogonality conditions

$$\int_0^{\infty} x^v L_n^{(\alpha)}(x) e^{-x} x^{\alpha}\, dx = 0, \quad 0 \leqslant v \leqslant n-1. \tag{10}$$

Eq. (10) and $L_n^{(\alpha)}(0) = 1$ imply (9). And the explicit representation of the Laguerre polynomials is

$$\frac{L_n^{(\alpha)}(x)}{\Gamma(\alpha+1)} = \sum_{v=0}^{n} \frac{n!}{(n-v)!} \frac{1}{\Gamma(\alpha+v+1)} \frac{(-x)^v}{v!}. \tag{11}$$

The Jacobi and Laguerre polynomials have the following asymptotic properties:

Theorem 4.1 (*The Mehler–Heine formulae*). *We have*

$$\lim_{n\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{z^2}{2n^2}\right) = \Gamma(\alpha+1)(z/2)^{-\alpha} J_{\alpha}(z), \tag{12}$$

$$\lim_{n\to\infty} L_n^{(\alpha)}\left(\frac{z}{n}\right) = \Gamma(\alpha+1) z^{-\alpha/2} J_{\alpha}(2z^{\frac{1}{2}}), \tag{13}$$

where $J_{\alpha}(z)$ is the Bessel function of the first kind, and the convergence is both uniform in every bounded region of the complex z-plane.

5. Some properties of $p_n^{(\alpha,\beta,\gamma)}(x)$

We have $\alpha, \beta, \gamma > -1$ and $a > 1$. The polynomials $p_n^{(\alpha,\beta,\gamma)}(x)$ of degree n have the following orthogonality condition:

$$\int_{-1}^{+1} x^v p_{2k}^{(\alpha,\beta,\gamma)}(x) |1-x|^{\alpha} (1+x)^{\beta} (a-x)^{\gamma}\, dx = 0, \quad 0 \leqslant v \leqslant k-1, \tag{14}$$

$$\int_{+1}^{\alpha} x^v p_{2k}^{(\alpha,\beta,\gamma)}(x) |1-x|^{\alpha} (1+x)^{\beta} (a-x)^{\gamma}\, dx = 0, \quad 0 \leqslant v \leqslant k-1, \tag{15}$$

$$\int_{-1}^{+1} x^v p_{2k+1}^{(\alpha,\beta,\gamma)}(x) |1-x|^{\alpha} (1+x)^{\beta} (a-x)^{\gamma}\, dx = 0, \quad 0 \leqslant v \leqslant k, \tag{16}$$

$$\int_{+1}^{\alpha} x^{\nu} p_{2k+1}^{(\alpha,\beta,\gamma)}(x)|1-x|^{\alpha}(1+x)^{\beta}(a-x)^{\gamma}\,dx = 0, \quad 0\leqslant \nu \leqslant k-1. \tag{17}$$

Furthermore, we set

$$p_n^{(\alpha,\beta,\gamma)}(1) = 1. \tag{18}$$

The polynomials $p_n^{(\alpha,\beta,\gamma)}(x)$ are uniquely determined by (14)–(18). We will give some important properties of these polynomials.

Lemma 5.1. *Suppose that* $|1-x| < |a-1|$, *then we have*

$$\begin{aligned}(a-x)^{\gamma} p_{2k}^{(\alpha,\beta,\gamma)}(x) \\ &= \binom{k+\alpha}{k}^{-1}(a-1)^{\gamma}\sum_{i=0}^{\infty}\binom{k+\gamma}{i}\binom{k+\alpha+i}{k}\left(\frac{1-x}{a-1}\right)^{i} P_k^{(\alpha+i,\beta)}(x).\end{aligned} \tag{19}$$

Proof. Using (14)–(15) we can prove the existence of c_k such that

$$\begin{aligned}&(1-x)^{\alpha}(1+x)^{\beta}(a-x)^{\gamma} p_{2k}^{(\alpha,\beta,\gamma)}(x) \\ &= c_k\left(\frac{d}{dx}\right)^{k}\{(1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma}\}.\end{aligned} \tag{20}$$

If $|1-x| < |a-1|$, we have

$$\begin{aligned}&\left(\frac{d}{dx}\right)^{k}\left[(1-x)^{k+\alpha}(1+x)^{k+\beta}\{(a-1)+(1-x)\}^{k+\gamma}\right] \\ &= \left(\frac{d}{dx}\right)^{k}\left[(1-x)^{k+\alpha}(1+x)^{k+\beta}(a-1)^{k+\gamma}\sum_{i=0}^{\infty}\binom{k+\gamma}{i}\left(\frac{1-x}{a-1}\right)^{i}\right] \\ &= (a-1)^{k+\gamma}\sum_{i=0}^{\infty}\binom{k+\gamma}{i}(a-1)^{-i}\left(\frac{d}{dx}\right)^{k}\left[(1-x)^{k+\alpha+i}(1+x)^{k+\beta}\right] \\ &= (1-x)^{\alpha}(1+x)^{\beta}(a-1)^{k+\gamma} \\ &\quad\times\sum_{i=0}^{\infty}\binom{k+\gamma}{i}\binom{k+\alpha+i}{k}(-1)^{k}2^{k}k!\left(\frac{1-x}{a-1}\right)^{i} P_k^{(\alpha+i,\beta)}(x).\end{aligned} \tag{21}$$

Thus we have

$$\begin{aligned}(a-x)^{\gamma} p_{2k}^{(\alpha,\beta,\gamma)}(x) &= c_k(a-1)^{k+\gamma}(-1)^{k}2^{k}k! \\ &\quad\times\sum_{i=0}^{\infty}\binom{k+\gamma}{i}\binom{k+\alpha+i}{k}\left(\frac{1-x}{a-1}\right)^{i} P_k^{(\alpha+i,\beta)}(x).\end{aligned} \tag{22}$$

Here by setting $x=1$, we obtain

$$c_k = \binom{k+\alpha}{k}^{-1}(a-1)^{-k}\frac{(-1)^{k}}{2^{k}k!}. \tag{23}$$

Therefore, we have

$$(a-x)^{\gamma} p_{2k}^{(\alpha,\beta,\gamma)}(x) = \binom{k+\alpha}{k}^{-1} (a-1)^{\gamma} \sum_{i=0}^{\infty} \binom{k+\gamma}{i} \binom{k+\alpha+i}{k} \left(\frac{1-x}{a-1}\right)^{i} P_k^{(\alpha+i,\beta)}(x). \quad (24)$$

We have thus proved the lemma. □

Lemma 5.2. *We have*

$$p_{2k+1}^{(\alpha,\beta,\gamma)}(x) = \frac{1+x}{2-(a-1)d_k} p_{2k}^{(\alpha,\beta+1,\gamma)}(x) - \frac{d_k(a-x)}{2-(a-1)d_k} p_{2k}^{(\alpha,\beta,\gamma+1)}(x), \quad (25)$$

where

$$d_k = \frac{\int_{-1}^{+1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1}(a-x)^{k+\gamma}\, dx}{\int_{-1}^{+1} (1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma+1}\, dx}. \quad (26)$$

Proof. There exist c_k' and c_k'' such that

$$(1-x)^{\alpha}(1+x)^{\beta}(a-x)^{\gamma} p_{2k+1}^{(\alpha,\beta,\gamma)}(x) = c_k' \left(\frac{d}{dx}\right)^k \{(1-x)^{k+\alpha}(1+x)^{k+\beta+1}(a-x)^{k+\gamma}\} + c_k'' \left(\frac{d}{dx}\right)^k \{(1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma+1}\} \quad (27)$$

and

$$c_k' \int_{-1}^{+1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1}(a-x)^{k+\gamma}\, dx + c_k'' \int_{-1}^{+1} (1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma+1}\, dx = 0. \quad (28)$$

From (27) and (19), we get

$$p_{2k+1}^{(\alpha,\beta,\gamma)}(x) = \frac{c_k'}{c_k}(1+x) p_{2k}^{(\alpha,\beta+1,\gamma)}(x) + \frac{c_k''}{c_k}(a-x) p_{2k}^{(\alpha,\beta,\gamma+1)}(x). \quad (29)$$

Since $p_{2k+1}^{(\alpha,\beta,\gamma)}(1) = 1$, we have

$$\frac{2c_k'}{c_k} + (a-1)\frac{c_k''}{c_k} = 1. \quad (30)$$

Therefore, if we set

$$d_k = \frac{\int_{-1}^{+1} (1-x)^{k+\alpha}(1+x)^{k+\beta+1}(a-x)^{k+\gamma}\, dx}{\int_{-1}^{+1} (1-x)^{k+\alpha}(1+x)^{k+\beta}(a-x)^{k+\gamma+1}\, dx}, \quad (31)$$

we obtain

$$\begin{cases} c'_k/c_k = 1/\{2-(a-1)d_k\}, \\ c''_k/c_k = -d_k/\{2-(a-1)d_k\}. \end{cases} \tag{32}$$

By (29) we obtain the desired result. □

Lemma 5.3. *Let d_k be defined as in Lemma 5.2. Put $x_0 = \frac{a-\sqrt{a^2+3}}{3}$. Then*

$$\lim_{k\to\infty} d_k = \frac{1+x_0}{a-x_0}. \tag{33}$$

This lemma is easily shown by the method of steepest descent (see [8, §8.71]).

6. A Mehler–Heine-type formula for $p_n^{(\alpha,\beta,\gamma)}$

We define certain Jacobi-like 2-orthogonal polynomials in the previous section. In this section we will prove an asymptotic formula for them:

Theorem 6.1 (*A Mehler–Heine formula for $p_n^{(\alpha,\beta,\gamma)}$*). *Let $p_n^{(\alpha,\beta,\gamma)}$ be the same as in Section 3.*

$$\lim_{n\to\infty} p_n^{(\alpha,\beta,\gamma)}\left(1-\frac{1-a}{3-a}\frac{2z^2}{n^2}\right) = \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z), \tag{34}$$

where the convergence is uniform in every bounded region of complex z-plane.

Proof. First let us consider $p_{2k}^{(\alpha,\beta,\gamma)}$. From (19) for sufficiently large k

$$\left(1+\frac{\frac{z^2}{2k^2}}{a-1}\right)^{\gamma} p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right) = \binom{k+\alpha}{k}^{-1}\sum_{i=0}^{\infty}\binom{k+\gamma}{i}\binom{k+\alpha+i}{k} \times\left(\frac{\frac{z^2}{2k^2}}{a-1}\right)^{i} P_k^{(\alpha+i,\beta)}\left(1-\frac{z^2}{2k^2}\right). \tag{35}$$

For the $(i+1)$st term of the right-hand side if z and i are fixed and $k\to\infty$, then we have

$$\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\gamma+1)}{\Gamma(i+1)\Gamma(k+\gamma-i+1)}\frac{\Gamma(k+\alpha+i+1)}{\Gamma(k+1)\Gamma(\alpha+i+1)} \times\left(\frac{\frac{z^2}{2k^2}}{a-1}\right)^{i} P_k^{(\alpha+i,\beta)}\left(1-\frac{z^2}{2k^2}\right) \longrightarrow \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha}\frac{(-1)^i}{i!}\left(\frac{2}{1-a}\frac{z}{2}\right)^{i} J_{\alpha+i}(z). \tag{36}$$

Here we used Theorem 4.1. Moreover, by the multiplication theorem for Bessel functions we obtain

$$\sum_{i=0}^{\infty} \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} \frac{(-1)^i}{i!}\left(\frac{2}{1-a}\frac{z}{2}\right)^i J_{\alpha+i}(z)$$
$$= \Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right). \tag{37}$$

Thus we need to show that we can interchange the limit and the summation in (35). We have the inequality

$$\left|\binom{k+\alpha}{k}^{-1}\binom{k+\gamma}{i}\binom{k+\alpha+i}{k}\left(\frac{\frac{z^2}{2k^2}}{a-1}\right)^i P_k^{(\alpha+i,\beta)}\left(1-\frac{z^2}{2k^2}\right)\right|$$
$$\leqslant \left|\frac{\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\gamma+1)}{\Gamma(k+\gamma-i+1)}\frac{\Gamma(k+\alpha+i+1)}{\Gamma(\alpha+i+1)i!}\right|\left|\frac{\frac{z^2}{2k^2}}{a-1}\right|^i$$
$$\times \sum_{v=0}^{k}\left|\frac{k!}{(k-v)!}\frac{\Gamma(k+\alpha+i+\beta+v+1)}{\Gamma(k+\alpha+i+\beta+1)}\frac{\left(-\frac{z^2}{4k^2}\right)^v}{\Gamma(\alpha+i+v+1)}\right|. \tag{38}$$

We denote the right-hand side by $f_{k,i}$. If $i \leqslant k+\lceil\gamma\rceil$, we have

$$\frac{\Gamma(k+\alpha+i+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\gamma+1)}{\Gamma(k+\gamma-i+1)k^{2i}}$$
$$= \frac{\prod_{j=1}^{i}(k+\alpha+j)}{k^i}\frac{\prod_{j=1}^{i}(k+\gamma-i+j)}{k^i}$$
$$\leqslant \left(1+\frac{\alpha+i}{k}\right)^i\left(1+\frac{\gamma}{k}\right)^i$$
$$\leqslant \left(1+\frac{1+\alpha+i}{k}\right)^{k+\lceil\gamma\rceil}\left(1+\frac{1+\gamma}{k}\right)^{k+\lceil\gamma\rceil}$$
$$= \mathcal{O}(e^i) \quad \text{as } k\to\infty, \tag{39}$$

uniformly in i, $0\leqslant i\leqslant k+\lceil\gamma\rceil$. And we have

$$\sum_{v=0}^{k}\left|\frac{k!}{(k-v)!}\frac{\Gamma(k+\alpha+i+\beta+v+1)}{\Gamma(k+\alpha+i+\beta+1)}\frac{\left(-\frac{z^2}{4k^2}\right)^v}{\Gamma(\alpha+i+v+1)}\right|$$
$$\leqslant \sum_{v=0}^{k} k^v\frac{(2k+i+\alpha+\beta)^v}{k^v(4k)^v}\frac{|z|^{2v}}{\Gamma(\alpha+v+1)} = \mathcal{O}(1), \tag{40}$$

as $k \to \infty$. Therefore, there exist constants m, m' such that

$$\sum_{i=0}^{k+\lceil\gamma\rceil} f_{k,i} \leqslant m' \sum_{i=0}^{k+\lceil\gamma\rceil} \frac{m^i}{\Gamma(\alpha+i+1)i!} = \mathcal{O}(1) \quad \text{as } k \to \infty. \tag{41}$$

Thus we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \binom{k+\alpha}{k}^{-1} \sum_{i=0}^{k+\lceil\gamma\rceil} \binom{k+\gamma}{i} \binom{k+\alpha+i}{k} \left(\frac{\frac{z^2}{2k^2}}{a-1}\right)^i P_k^{(\alpha+i,\beta)}\left(1-\frac{z^2}{2k^2}\right) \\
&\quad = \sum_{i=0}^{\infty} \lim_{k\to\infty} \binom{k+\alpha}{k}^{-1} \binom{k+\gamma}{i} \binom{k+\alpha+i}{k} \\
&\qquad \times \left(\frac{\frac{z^2}{2k^2}}{a-1}\right)^i P_k^{(\alpha+i,\beta)}\left(1-\frac{z^2}{2k^2}\right) \\
&\quad = \Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right).
\end{aligned} \tag{42}$$

On the other hand we have

$$\begin{aligned}
&\left| \frac{k!}{(k-v)!} \frac{\Gamma(k+\alpha+i+1+\beta+v+1)}{\Gamma(k+\alpha+i+1+\beta+1)} \frac{\left(-\frac{z^2}{4k^2}\right)^v}{\Gamma(\alpha+i+1+v+1)} \right| \\
&\quad \leqslant \frac{\Gamma(2k+\alpha+i+\beta+1)}{\Gamma(k+\alpha+i+\beta+1)\Gamma(\alpha+i+1)} \\
&\qquad \times \left| \frac{k!}{(k-v)!} \frac{\Gamma(k+\alpha+i+\beta+v+1)}{\Gamma(k+\alpha+i+\beta+1)} \frac{\left(-\frac{z^2}{4k^2}\right)^v}{\Gamma(\alpha+i+v+1)} \right|.
\end{aligned} \tag{43}$$

Thus for $i \geqslant k + \lceil\gamma\rceil$ and a sufficiently large k, we obtain

$$\begin{aligned}
\frac{f_{k,i+1}}{f_{k,i}} &\leqslant \frac{i-k-\gamma}{i+1} \frac{k+\alpha+i+1}{\alpha+i+1} \frac{2k+\alpha+i+\beta+1}{(k+\alpha+i+\beta+1)(\alpha+i+1)} \left| \frac{\frac{z^2}{2k^2}}{a-1} \right| \\
&\leqslant \frac{6}{\alpha+i+1} \left| \frac{\frac{z^2}{2k^2}}{a-1} \right| < c < 1.
\end{aligned} \tag{44}$$

Therefore, we obtain an inequality

$$\sum_{i=k+\lceil\gamma\rceil}^{\infty} f_{k,i} < f_{k,k+\lceil\gamma\rceil} \frac{1}{1-c}. \tag{45}$$

By (41), we know $\lim_{k\to\infty} f_{k,k+\lceil\gamma\rceil} = 0$. Hence we obtain

$$\lim_{k\to\infty} \sum_{i=k+\lceil\gamma\rceil}^{\infty} f_{k,i} = 0. \tag{46}$$

Thus from (42) and (46) we conclude

$$\lim_{k\to\infty} p_{2k}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right) = \Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right). \tag{47}$$

Next let us consider $p_{2k+1}^{(\alpha,\beta,\gamma)}$. Lemmas 5.2 and 5.3 lead us to the following asymptotic behavior:

$$\begin{aligned}
&p_{2k+1}^{(\alpha,\beta,\gamma)}\left(1-\frac{z^2}{2k^2}\right)\\
&\quad= \frac{2-\frac{z^2}{2k^2}}{2-(a-1)d_k} p_{2k}^{(\alpha,\beta+1,\gamma)}\left(1-\frac{z^2}{2k^2}\right)\\
&\qquad -\frac{d_k\left(a-1+\frac{z^2}{2k^2}\right)}{2-(a-1)d_k} p_{2k}^{(\alpha,\beta,\gamma+1)}\left(1-\frac{z^2}{2k^2}\right)\\
&\quad\longrightarrow \frac{2}{2-(a-1)\frac{1+x_0}{a-x_0}}\Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right)\\
&\qquad -\frac{\frac{1+x_0}{a-x_0}(a-1)}{2-(a-1)\frac{1+x_0}{a-x_0}}\Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right)\\
&\quad= \Gamma(\alpha+1)\left(\sqrt{\frac{3-a}{1-a}}\frac{z}{2}\right)^{-\alpha} J_\alpha\left(\sqrt{\frac{3-a}{1-a}}z\right). \qquad \square
\end{aligned} \tag{48}$$

7. Some properties of $l_n^{(\alpha,\beta)}(x)$

We have $\alpha, \beta > -1$ and $b < 0$. The polynomials $l_n^{(\alpha,\beta)}(x)$ of degree n have the following orthogonality condition:

$$\int_0^\infty x^\nu l_{2k}^{(\alpha,\beta)}(x)e^{-x}|x|^\alpha(x-b)^\beta\,dx = 0, \quad 0\leqslant \nu\leqslant k-1, \tag{49}$$

$$\int_b^0 x^\nu l_{2k}^{(\alpha,\beta)}(x)e^{-x}|x|^\alpha(x-b)^\beta\,dx = 0, \quad 0\leqslant \nu\leqslant k-1, \tag{50}$$

$$\int_0^\infty x^v l_{2k+1}^{(\alpha,\beta)}(x)e^{-x}|x|^\alpha(x-b)^\beta\,dx = 0, \quad 0\leqslant v\leqslant k, \tag{51}$$

$$\int_b^0 x^v l_{2k+1}^{(\alpha,\beta)}(x)e^{-x}|x|^\alpha(x-b)^\beta\,dx = 0, \quad 0\leqslant v\leqslant k-1. \tag{52}$$

If we set

$$l_n^{(\alpha,\beta)}(0) = 1, \tag{53}$$

the polynomials $l_n^{(\alpha,\beta)}$ are uniquely determined.

Lemma 7.1. *Suppose that* $|x| < |b|$, *then we have*

$$\begin{aligned}&(x-b)^\beta l_{2k}^{(\alpha,\beta)}(x)\\ &\quad= \binom{k+\alpha}{k}^{-1}(-b)^\beta \sum_{i=0}^\infty \binom{k+\beta}{i}\binom{k+\alpha+i}{k}\left(-\frac{x}{b}\right)^i L_k^{(\alpha+i)}(x).\end{aligned}$$

Proof. There exists a constant $C_k^{(\alpha)}$ such that

$$e^{-x}x^\alpha(x-b)^\beta l_{2k}^{(\alpha,\beta)}(x) = C_k^{(\alpha)}\left(\frac{d}{dx}\right)^k \left\{e^{-x}x^{k+\alpha}(x-b)^{k+\beta}\right\}. \tag{54}$$

Since $|x| < |b|$, we have

$$\begin{aligned}&e^{-x}x^\alpha(x-b)^\beta l_{2k}^{(\alpha,\beta)}(x)\\ &\quad= C_k^{(\alpha)}\left(\frac{d}{dx}\right)^k\left\{e^{-x}x^{k+\alpha}(-b)^{k+\beta}\sum_{i=0}^\infty\binom{k+\beta}{i}\left(-\frac{x}{b}\right)^i\right\}\\ &\quad= C_k^{(\alpha)}(-b)^{k+\beta}\sum_{i=0}^\infty\binom{k+\beta}{i}\binom{k+\alpha+i}{k}\left(-\frac{1}{b}\right)^i k!e^{-x}x^{\alpha+i}L_k^{(\alpha+i)}(x).\end{aligned}$$

As $l_{2k}^{(\alpha,\beta)}(0) = 1$, we get

$$C_k^{(\alpha)} = \binom{k+\alpha}{k}^{-1}\frac{(-b)^{-k}}{k!}. \tag{55}$$

We have thus proved the lemma. □

Lemma 7.2. *We have*

$$l_{2k+1}^{(\alpha,\beta)}(x) = -\frac{k+\alpha+1}{\alpha+1}D_k\; x\; l_{2k}^{(\alpha+1,\beta)}(x) + \frac{x-b}{-b}l_{2k}^{(\alpha,\beta+1)}(x), \tag{56}$$

where

$$D_k = \frac{\int_0^\infty e^{-x}x^{k+\alpha+1}(x-b)^{k+\beta}\,dx}{\int_0^\infty e^{-x}x^{k+\alpha}(x-b)^{k+\beta+1}\,dx}. \tag{57}$$

Proof. There exist constants C'_k and C''_k such that

$$\begin{aligned} e^{-x}x^{\alpha}(x-b)^{\beta}l_{2k+1}^{(\alpha,\beta)}(x) &= C'_k\left(\frac{d}{dx}\right)^k\left\{e^{-x}x^{k+\alpha+1}(x-b)^{k+\beta}\right\} \\ &\quad + C''_k\left(\frac{d}{dx}\right)^k\left\{e^{-x}x^{k+\alpha}(x-b)^{k+\beta+1}\right\} \\ &= \frac{C'_k}{C_k^{(\alpha+1)}}e^{-x}x^{\alpha+1}(x-b)^{\beta}l_{2k}^{(\alpha+1,\beta)}(x) \\ &\quad + \frac{C''_k}{C_k^{(\alpha)}}e^{-x}x^{\alpha}(x-b)^{\beta+1}l_{2k}^{(\alpha,\beta+1)}(x), \end{aligned} \tag{58}$$

and

$$C'_k\int_0^{\infty}e^{-x}x^{k+\alpha}(x-b)^{k+\beta+1}\,dx + C''_k\int_0^{\infty}e^{-x}x^{k+\alpha+1}(x-b)^{k+\beta}\,dx = 0. \tag{59}$$

As $l_{2k+1}^{(\alpha,\beta)}(0) = 1$, we have

$$C''_k = \frac{C_k^{(\alpha)}}{-b} = \binom{k+\alpha}{k}^{-1}\frac{(-b)^{-k-1}}{k!}. \tag{60}$$

Put

$$D_k = \frac{\int_0^{\infty}e^{-x}x^{k+\alpha+1}(x-b)^{k+\beta}\,dx}{\int_0^{\infty}e^{-x}x^{k+\alpha}(x-b)^{k+\beta+1}\,dx}, \tag{61}$$

then we have

$$l_{2k+1}^{(\alpha,\beta)}(x) = -\frac{k+\alpha+1}{\alpha+1}D_k\; x\; l_{2k}^{(\alpha+1,\beta)}(x) + \frac{x-b}{-b}l_{2k}^{(\alpha,\beta+1)}(x). \qquad \square \tag{62}$$

Lemma 7.3.

$$\lim_{k\to\infty} D_k = 1. \tag{63}$$

We can easily show this lemma by the method of steepest descent (see [8, §8.72]).

8. A Mehler–Heine-type formula for $l_n^{(\alpha,\beta)}$

We also give a Mehler–Heine formula for $l_n^{(\alpha,\beta)}$.

Theorem 8.1. *We have*

$$\lim_{n\to\infty} l_n^{(\alpha,\beta)}\left(\frac{4bz^2}{n^2}\right) = \Gamma(\alpha+1)z^{-\alpha}J_{\alpha}(z) \tag{64}$$

and this convergence is uniform in every bounded region of complex z-plane.

Proof. From Lemma 7.1 we have for sufficiently large k

$$\begin{aligned}
&\left(\frac{bz^2}{k^2}-b\right)^{\beta} l_{2k}^{(\alpha,\beta)}\left(\frac{bz^2}{k^2}\right)\\
&\quad =\Gamma(\alpha+1)(-b)^{\beta}\sum_{i=0}^{\infty}\frac{\Gamma(k+\alpha+i+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta-i+1)}\\
&\qquad \times\frac{(-z^2/k^2)^i}{i!\,\Gamma(\alpha+i+1)}L_k^{(\alpha+i)}\left(\frac{bz^2}{k^2}\right).
\end{aligned}$$

We have

$$\begin{aligned}
&\left|\sum_{i=0}^{\infty}\frac{\Gamma(k+\alpha+i+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta-i+1)}\frac{(-z^2/k^2)^i}{i!\,\Gamma(\alpha+i+1)}L_k^{(\alpha+i)}\left(\frac{bz^2}{k^2}\right)\right|\\
&\quad\leqslant\sum_{i=0}^{\infty}\left|\frac{\Gamma(k+\alpha+i+1)}{k^i\,\Gamma(k+\alpha+1)}\frac{\Gamma(k+\beta+1)}{k^i\,\Gamma(k+\beta-i+1)}\frac{(-z^2)^i}{i!}\right|\\
&\qquad\times\sum_{v=0}^{k}\left|\frac{k^v}{\Gamma(\alpha+i+1)}\frac{(bz^2)^v}{k^{2v}v!}\right|\\
&\quad=\sum_{i=0}^{\infty}\left|\prod_{j=1}^{i}\left(1+\frac{\alpha+j}{k}\right)\prod_{j=1}^{i}\left(1+\frac{\beta-i+j}{k}\right)\frac{(-z^2)^i}{i!}\right|\\
&\qquad\times\sum_{v=0}^{k}\left|\frac{1}{\Gamma(\alpha+i+1)}\frac{(bz^2)^v}{k^v v!}\right|.
\end{aligned}$$

Put

$$\begin{aligned}
F_{k,i}=&\left|\prod_{j=1}^{i}\left(1+\frac{\alpha+j}{k}\right)\prod_{j=1}^{i}\left(1+\frac{\beta-i+j}{k}\right)\frac{(-z^2)^i}{i!}\right|\\
&\times\sum_{v=0}^{k}\left|\frac{1}{\Gamma(\alpha+i+1)}\frac{(bz^2)^v}{k^v v!}\right|.
\end{aligned}\tag{65}$$

If $i\leqslant k+\lceil\beta\rceil$, we have

$$\begin{aligned}
F_{k,i}&\leqslant\frac{1}{\Gamma(\alpha+1)}\left|\left(1+\frac{\alpha+i}{k}\right)^i\left(1+\frac{\beta}{k}\right)^i\frac{(-z^2)^i}{i!}\right|\sum_{v=0}^{k}\left|\frac{(bz^2)^v}{v!}\right|\\
&\leqslant\frac{1}{\Gamma(\alpha+1)}\left|\left(1+\frac{\alpha+1+i}{k}\right)^{k+\lceil\beta\rceil}\left(1+\frac{\beta+1}{k}\right)^{k+\lceil\beta\rceil}\frac{(-z^2)^i}{i!}\right|e^{|bz^2|}\\
&\leqslant M'\frac{M^i}{i!},
\end{aligned}\tag{66}$$

where M and M' are independent of i and k. If $i \geqslant k + \lceil\beta\rceil$ and k is sufficiently large, we have

$$\begin{aligned}\frac{F_{k,i+1}}{F_{k,i}} &\leqslant \frac{1}{\alpha+i+1}\left|\left(1+\frac{\alpha+i+1}{k}\right)\left(1+\frac{\beta-i}{k}\right)\frac{-z^2}{i+1}\right| \\ &\leqslant \frac{6}{k^2}|z|^2 < C < 1.\end{aligned}$$

Therefore, we obtain

$$\sum_{i=k+\lceil\beta\rceil}^{\infty} F_{k,i} < F_{k,k+\lceil\beta\rceil}\frac{1}{1-C} \leqslant M' \frac{M^{k+\lceil\beta\rceil}}{(k+\lceil\beta\rceil)!}\frac{1}{1-C} \longrightarrow 0 \quad \text{as } k \to \infty.$$

As $k \to \infty$,

$$\begin{aligned}&\left(\frac{bz^2}{k^2}-b\right)^{\beta} l_{2k}^{(\alpha,\beta)}\left(\frac{bz^2}{k^2}\right) \\ &= \Gamma(\alpha+1)(-b)^{\beta}\sum_{i=0}^{k+\lceil\beta\rceil}\frac{\Gamma(k+\alpha+i+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta-i+1)} \\ &\quad \times \frac{(-z^2/k^2)^i}{i!\,\Gamma(\alpha+i+1)} L_k^{(\alpha+i)}\left(\frac{bz^2}{k^2}\right) + o(1) \\ &\longrightarrow \Gamma(\alpha+1)(-b)^{\beta}\sum_{i=0}^{\infty}\lim_{k\to\infty}\frac{\Gamma(k+\alpha+i+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta-i+1)} \\ &\quad \times \frac{(-z^2/k^2)^i}{i!\,\Gamma(\alpha+i+1)} L_k^{(\alpha+i)}\left(\frac{bz^2}{k^2}\right) \\ &= \Gamma(\alpha+1)(-b)^{\beta}\sum_{i=0}^{\infty}\frac{(-z^2)^i}{i!\,\Gamma(\alpha+i+1)} \\ &= \Gamma(\alpha+1)(-b)^{\beta} z^{-\alpha} J_{\alpha}(z).\end{aligned}$$

From Lemmas 7.2 and 7.3 we have

$$\begin{aligned} l_{2k+1}^{(\alpha,\beta)}\left(\frac{bz^2}{k^2}\right) &= -\frac{k+\alpha+1}{\alpha+1} D_k\left(\frac{bz^2}{k^2}\right) l_{2k}^{(\alpha+1,\beta)}\left(\frac{bz^2}{k^2}\right) \\ &\quad + \frac{\left(\frac{bz^2}{k^2}\right)-b}{-b} l_{2k}^{(\alpha,\beta+1)}\left(\frac{bz^2}{k^2}\right) \\ &\longrightarrow \lim_{k\to\infty} l_{2k}^{(\alpha,\beta+1)}\left(\frac{bz^2}{k^2}\right) = \Gamma(\alpha+1) z^{-\alpha} J_{\alpha}(z) \quad \text{as } k \to \infty. \end{aligned} \tag{67}$$

Therefore we have proved the theorem. □

9. A note on Theorems 6.1 and 8.1

From Theorems 6.1 and 8.1 we can obtain an interesting theorem immediately.

Theorem 9.1. *We set the positive zeros of* J_α *be* $j_1 < j_2 < \cdots$ *and the zeros of* $p_n^{(\alpha,\beta,\gamma)}(z)$, $l_n^{(\alpha,\beta)}(z)$ *be as follows*:

$$p_n^{(\alpha,\beta,\gamma)} : \begin{cases} -1 < s_k^{(n)} < \cdots < s_1^{(n)} < 1 < t_1^{(n)} < \cdots < t_k^{(n)} < a \\ \quad \text{if } n = 2k, \\ \\ -1 < s_{k+1}^{(n)} < \cdots < s_1^{(n)} < 1 < t_1^{(n)} < \cdots < t_k^{(n)} < a \\ \quad \text{if } n = 2k+1, \end{cases} \tag{68}$$

$$l_n^{(\alpha,\beta)} : \begin{cases} b < u_k^{(n)} < \cdots < u_1^{(n)} < 0 < v_1^{(n)} < \cdots < v_k^{(n)} & \text{if } n = 2k, \\ b < u_{k+1}^{(n)} < \cdots < u_1^{(n)} < 0 < v_1^{(n)} < \cdots < v_k^{(n)} & \text{if } n = 2k+1. \end{cases} \tag{69}$$

Then for any i

$$\begin{cases} \left| n\sqrt{\frac{3-a}{a-1}\frac{s_i^{(n)}-1}{2}} \right| \longrightarrow \infty, \quad n\sqrt{\frac{3-a}{a-1}\frac{t_i^{(n)}-1}{2}} \longrightarrow j_i & \text{if } 1 < a < 3, \\ n\sqrt{\frac{3-a}{a-1}\frac{s_i^{(n)}-1}{2}} \longrightarrow j_i, \quad \left| n\sqrt{\frac{3-a}{a-1}\frac{t_i^{(n)}-1}{2}} \right| \longrightarrow \infty & \text{if } 3 < a, \end{cases} \tag{70}$$

$$\frac{n}{2}\sqrt{\frac{u_i^{(n)}}{b}} \longrightarrow j_i, \left| \frac{n}{2}\sqrt{\frac{v_i^{(n)}}{b}} \right| \longrightarrow \infty \tag{71}$$

as $n \to \infty$.

This theorem can be interpreted as the phenomena of charge repelling in the following electrostatics model. Fix three point charges with magnitudes $\frac{\alpha+1}{2}$, $\frac{\beta+1}{2}$ and $\frac{\gamma+1}{2}$ on the line at positions $x = +1$, $x = -1$ and $x = a > 1$, respectively. Suppose $n = 2k$ movable unit charges are located at distinct points $-1 < x_k^{(n)} < \cdots < x_1^{(n)} < +1$ and $+1 < y_1^{(n)} < \cdots < y_k^{(n)} < a$ on the line, and that all $n+3$ charges repel with an interaction force derived from a logarithmic potential. Then the equilibrium position of the movable unit charges is attained at the zeroes $p_{2n}^{(\alpha,\beta,\gamma)}(x_i) = 0$, $i = 1, \ldots, 2n$. For more information see [4].

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